

Partition functions and double-trace deformations in AdS/CFT

Danilo E. Díaz and Harald Dorn

Humboldt-Universität zu Berlin, Institut für Physik

Newtonstr.15, D-12489 Berlin

E-mail: ddiaz@physik.hu-berlin.de, dorn@physik.hu-berlin.de

ABSTRACT: We study the effect of a relevant double-trace deformation on the partition function (and conformal anomaly) of a CFT_d at large N and its dual picture in AdS_{d+1} . Three complementary previous results are brought into *full* agreement with each other: bulk [1] and boundary [2] computations, as well as their formal identity [3]. We show the *exact* equality between the dimensionally regularized partition functions or, equivalently, fluctuation determinants involved. A series of results then follows: (i) equality between the renormalized partition functions for all d ; (ii) for *all* even d , correction to the conformal anomaly; (iii) for even d , the mapping entails a mixing of UV and IR effects on the same side (bulk) of the duality, with no precedent in the leading order computations; and finally, (iv) a subtle relation between overall coefficients, volume renormalization and IR-UV connection. All in all, we get a clean test of the AdS/CFT correspondence beyond the classical SUGRA approximation in the bulk and at subleading $O(1)$ order in the large- N expansion on the boundary.

KEYWORDS: AdS-CFT Correspondence, Anomalies in Field and String Theories, $1/N$ Expansion.

Contents

1. Introduction	1
2. The bulk computation: one-loop effective action	4
2.1 Dimensionally regularized volume	5
2.2 Dimensionally regularized one-loop effective potential	6
2.3 Dimensionally regularized one-loop effective action	7
3. The boundary computation: deformed partition function	7
4. Back to the physical dimensions	9
4.1 $d=\text{odd}$: renormalized partition functions	9
4.2 $d=\text{even}$: anomaly and renormalized partition functions	10
5. Miscellaneous comments	11
6. Conclusion	12
A. Convolution of bulk-to-boundary propagators	14
B. Useful formulas	15

1. Introduction

Maldacena's conjecture and its calculational prescription [4] entail the equality between the partition function of String/M-theory (with prescribed boundary conditions) in the product space $AdS_{d+1} \times X$, where X is certain compact manifold, and the generating functional of the boundary CFT_d . It has been fairly well tested at the level of classical SUGRA in the bulk and at the corresponding leading order at large N on the boundary.

One of the most remarkable tests is the mapping of the conformal anomaly [5]. Since the rank N of the group measures the size of the geometry in Planck units, quantum corrections correspond to subleading terms in the large N limit. Corrections of order $O(N)$ have also been obtained [6], but they rather correspond to tree-level corrections after inclusion of open or unoriented closed strings. Truly quantum corrections face the notorious difficulty of RR-backgrounds and only few examples, besides semiclassical limits of the correspondence, have circumvented it and corroborated the conjecture at this nontrivial level [7]. These results rely on whole towers of KK-states and SUSY. The regimes in which the bulk and boundary computations can be done do not overlap and some sort of non-renormalization must be invoked.

In this note we deal with a universal AdS/CFT result, not relying on SUSY or any other detail encoded in the compact space X , concerning an $O(1)$ correction to the conformal anomaly under a flow produced by a double-trace deformation. This was first computed in the bulk of AdS [1] and confirmed shortly after by a field theoretic computation on the dual boundary theory [2] (see also [8]).

Let us roughly recapitulate the sequence of developments leading to this remarkable success. It starts with a scalar field ϕ with “tachyonic” mass in the window $-\frac{d^2}{4} \leq m^2 < -\frac{d^2}{4} + 1$ where two AdS-invariant quantizations are known to exist [9]. The conformal dimensions of the dual CFT operators, given by the two roots Δ_+ and Δ_- of the AdS/CFT relation $m^2 = \Delta(\Delta - d)$:

$$\Delta_{\pm} = \frac{d}{2} \pm \nu, \quad \nu = \sqrt{\frac{d^2}{4} + m^2} \tag{1.1}$$

are then ($0 \leq \nu < 1$) both above the unitarity bound. The modern AdS/CFT interpretation [10] assigns the same bulk theory to two different CFTs at the boundary, whose generating functionals are related to each other by Legendre transformation at leading large N . The only difference is the interchange of the roles of boundary operator/source associated to the asymptotic behavior of the bulk scalar field near the conformal boundary. The whole picture fits into the generalized AdS/CFT prescription to incorporate boundary multi-trace operators [11]. The two CFTs are then the end points of a RG flow triggered by the relevant perturbation $f O_{\alpha}^2$ of the α -CFT, where the operator O_{α} has dimension Δ_- (so that $\Delta_- + \Delta_+ = d$, $\Delta_- \leq \Delta_+ \Rightarrow 2\Delta_- \leq d$). The α -theory flows into the β -theory which now has an operator O_{β} with dimension $\Delta_+ = d - \Delta_-$, conjugate to Δ_- . The rest of the operators remains untouched at leading large N , which suggests that the metric and the rest of the fields involved should retain their background values, only the dual bulk scalar changes its asymptotics.¹

The crucial observation in [1] is the following: since the only change in the bulk is in the asymptotics of the scalar field, the effect on the partition function cannot be seen at the classical gravity level in the bulk, i.e. at leading large N , since the background solution has $\phi = 0$; but the quantum fluctuations around this solution, given by the functional determinant of the kinetic term (inverse propagator), are certainly sensitive to the asymptotics since there are two different propagators G_{Δ} corresponding to the two different AdS-invariant quantizations. The partition function including the one-loop correction is

$$Z_{\text{grav}}^{\pm} = Z_{\text{grav}}^{\text{class}} \cdot [\det_{\pm}(-\square + m^2)]^{-\frac{1}{2}}, \tag{1.2}$$

where $Z_{\text{grav}}^{\text{class}}$ refers to the usual saddle point approximation. Notice that the functional determinant is independent of N , this makes the scalar one-loop quantum correction an $O(1)$ effect. The 1-loop computation turns out to be very simple for even dimension d and is given by a polynomial in Δ . No infinities besides the IR one, related to the volume of AdS, show up in the relative change $Z_{\text{grav}}^+ / Z_{\text{grav}}^-$, since the UV-divergences can be controlled exactly in the same way for both propagators. From this correction to the

¹The simplest realization of this behavior being the $O(N)$ vector model in $2 < d < 4$, see e.g. [12, 13].

classical gravitational action one can read off an $O(1)$ contribution to the *holographic conformal anomaly* [5].

The question whether this $O(1)$ correction to the anomaly could be recovered from a pure CFT_d calculation was answered shortly after in the affirmative [2]. Using the Hubbard-Stratonovich transformation (or auxiliary field trick) and large N factorization of correlators, the Legendre transformation relation at leading large N is shown. An extra $O(1)$ contribution, the fluctuation determinant of the auxiliary field, is also obtained. Turning the sources to zero, the result for the CFT partition function can be written as

$$Z_\beta = Z_\alpha \cdot [\det(\Xi)]^{-\frac{1}{2}}, \tag{1.3}$$

where the kernel $\Xi \equiv \mathbb{I} + f \langle O_\alpha O_\alpha \rangle$ in position space in \mathbb{R}^d is given by $\delta^d(x, x') + f \frac{C_{\Delta_-}}{|x-x'|^{2\Delta_-}}$. The β -CFT is reached in the limit $f \rightarrow \infty$.

From the CFT point of view, the conformal invariance of this functional determinant has then to be probed. Putting the theory on the sphere \mathbb{S}^d and expanding in spherical harmonics, using Stirling formula for large orbital quantum number l and zeta-function regularization, the coefficient of the log-divergent term is isolated. It happily coincides (for the explored cases $d = 2, 4, 6, 8$) with the AdS_{d+1} prediction for the anomaly.

Despite the successful agreement, there are several issues in this derivation that ought to be further examined. No track is kept on the overall coefficient in the CFT computation, in contrast to the mapping at leading order [5] that matches the overall coefficient as well. For odd dimension d , the CFT determinant has no anomaly, whereas there is a nonzero AdS result that could be some finite term in field theory not computed so far [2]. From a computational point of view the results are quite different. The AdS answer is a polynomial for generic even dimension, whereas for odd d only numerical results are reported. The CFT answer, on the other hand, is obtained for few values of the dimension d , a proof for generic d is lacking. Yet, the very same $O(1)$ nature of the correction on both sides of the correspondence calls for a full equivalence between the relative change in the partition functions, and not only just the conformal anomaly. This poses a new challenge since in the above derivation there are several (divergent) terms that were disregarded, for they do not contribute to the anomaly.

As we have seen, it all boils down to computing functional determinants. In a more recent work [3], a “kinematical” understanding of the agreement between the bulk and boundary computations was achieved based on the equality between the determinants. The key is to explicitly separate the transverse coordinates in AdS, expand the bulk determinant in this basis inserting the eigenvalues of the transverse Laplacian weighted with their degeneracies. In this way, one gets a weighted sum/integral of effective radial (one-dimensional) determinants which are then evaluated via a suitable generalization of the Gel’fand-Yaglom formula. The outcome turns out to coincide with the expansion of the auxiliary field fluctuation determinant of [2]. However, this procedure is known to be rather formal (see, e.g., [14] and references therein) and the result to be certainly divergent. No further progress is done on either side of this *formal* equality and the issue of reproducing the full bulk result from a field theoretic computation at the boundary remains open.

We will show that all above open questions can be thoroughly clarified or bypassed if one uses dimensional regularization to control all the divergences. Both IR and UV divergencies are now on equal footing, which is precisely the essence of the IR-UV connection [15]: the key to the holographic bound is that an IR regulator for the boundary area becomes an UV regulator in the dual CFT. The bulk effective potential times the infinite AdS volume, i.e. the effective action, and the boundary sum, using Gauß’s “proper-time representation” for the digamma function to perform it, are shown to coincide in dimensional regularization.

The paper is organized as follows: we start with the bulk partition function and compute the regularized effective action. Here one needs to compute separately the volume and the effective potential. Then we move to the boundary to compute the change induced by the double-trace deformation. Having established the equivalence for dimensionally regularized quantities we go back to the physical dimensions and extract the relevant results for the renormalized partition functions and the conformal anomaly. Before the final conclusions, the paper still has a section with some further technical remarks. Some useful relations and formulas are collected in two appendices.

2. The bulk computation: one-loop effective action

Let us start with the Euclidean action for gravity and a scalar field

$$S_{d+1}^{\text{class}} = \frac{-1}{2\kappa^2} \int d\text{vol}_{d+1} [R - \Lambda] + \int d\text{vol}_{d+1} \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right]. \quad (2.1)$$

For negative Λ the Euclidean version of AdS_{d+1} , i.e. the Lobachevsky space \mathbb{H}^{d+1} , is a classical solution. There are, of course, additional terms like the Gibbons-Hawking surface term² and contributions from other fields, but they will play no role in what follows, nor will the details of the leading large N duality. This is an indication of universality of the results.

We are interested in the quantum one-loop correction from the scalar field with the Δ_+ or Δ_- asymptotic behavior

$$S_{d+1}^{\pm} = \frac{1}{2} \log \det_{\pm}(-\square + m^2) = \frac{1}{2} \text{tr}_{\pm} \log(-\square + m^2). \quad (2.2)$$

It will prove simpler to consider instead the quantities

$$\frac{\partial}{\partial m^2} S_{d+1}^{\pm} = \frac{1}{2} \text{tr}_{\pm} \frac{1}{-\square + m^2}. \quad (2.3)$$

This rather symbolic manipulation, casted into a concrete form, reads

$$\frac{\partial}{\partial m^2} S_{d+1}^{\pm} = \frac{1}{2} \int d\text{vol}_{\mathbb{H}^{d+1}} G_{\Delta_{\pm}}(z, z). \quad (2.4)$$

²Additional covariant local boundary counterterms that render the on-shell classical bulk action finite are obtained by the ‘holographic renormalization’ procedure, see e.g. [16].

There are two kinds of divergencies here, one is the infinite volume of the hyperbolic space (IR) and the other is the short distance singularity of the propagator (UV). The latter is conventionally controlled by taking the difference of the \pm -versions; this produces a finite result and was the crucial observation in [1]. Then one gets for the difference of the one-loop corrections for the Δ_{\pm} asymptotics

$$\frac{\partial}{\partial m^2} (S_{d+1}^+ - S_{d+1}^-) = \frac{1}{2} \int d\text{vol}_{\mathbb{H}^{d+1}} \{G_{\Delta_+}(z, z) - G_{\Delta_-}(z, z)\} . \quad (2.5)$$

One might be tempted to factorize away the volume (usual procedure) and work further only with the effective potential. However, the perfect matching with the boundary computation will require keeping track of the volume as well. In the spirit of the IR-UV connection we now use dimensional regularization to control both the IR divergencies in the bulk as well as the UV divergencies on the boundary.

2.1 Dimensionally regularized volume

Starting from the usual representation of \mathbb{H}^{d+1} in terms of a unit ball with metric $ds^2 = 4(1 - x^2)^{-2} dx^2$ one gets, after the substitution $r = (1 - |x|)/(1 + |x|)$, the metric

$$G = r^{-2}[(1 - r^2)^2 g_0 + dr^2], \quad (2.6)$$

with $4g_0$ being the usual round metric on \mathbb{S}^d . Then

$$\left(\frac{\det G}{\det g_0}\right)^{\frac{1}{2}} = r^{-1-d} (1 - r^2)^d, \quad (2.7)$$

and the volume is then given by

$$\int d\text{vol}_{\mathbb{H}^{d+1}} = 2^{-d} \text{vol}_{\mathbb{S}^d} \int_0^1 dr r^{-d-1} (1 - r^2)^d. \quad (2.8)$$

Up to this point, we have just followed [17] to compute the volume. From here on, there are two standard ways to proceed in the mathematical literature, namely Hadamard or Riesz regularization (see, e.g. [18]). We will use none of them, although our choice of dimensional regularization is closer to Riesz's scheme. This IR-divergent volume will now be controlled with DR:³ set $d \rightarrow D = d - \epsilon$ and perform the integral to get, after some manipulations,

$$\text{vol}_{\mathbb{H}^{D+1}} = \pi^{\frac{D}{2}} \Gamma\left(-\frac{D}{2}\right). \quad (2.9)$$

Let us now send ϵ to zero:

$$\text{vol}_{\mathbb{H}^{D+1}} = \frac{\mathcal{L}_{d+1}}{\epsilon} + \mathcal{V}_{d+1} + o(1). \quad (2.10)$$

³Some virtues of dimensional regularization in relation with holographic anomalies were already apparent in [19, 20].

For *even* d we find the “integrated conformal anomaly” (integral of Branson’s Q-curvature, a generalization of the scalar curvature, see e.g. [17]) and renormalized volume given by

$$\mathcal{L}_{d+1} = (-1)^{\frac{d}{2}} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d+2}{2}\right)}, \tag{2.11}$$

$$\mathcal{V}_{d+1} = \frac{1}{2} \mathcal{L}_{d+1} \cdot \left[\psi\left(1 + \frac{d}{2}\right) - \log \pi \right]. \tag{2.12}$$

For *odd* d in turn, \mathcal{L}_{d+1} vanishes and the renormalized volume is given by

$$\mathcal{V}_{d+1} = (-1)^{\frac{d+1}{2}} \frac{\pi^{\frac{d+2}{2}}}{\Gamma\left(\frac{d+2}{2}\right)}. \tag{2.13}$$

The *conformal invariants* \mathcal{L}_{d+1} for $d = \text{even}$ and \mathcal{V}_{d+1} for $d = \text{odd}$ coincide with those obtained by Hadamard regularization [17]. For even d in turn, the regularized volume fails to be conformal invariant and its integrated infinitesimal variation under a change of representative metric on the boundary is precisely given by \mathcal{L} [17]. In all, the presence of the pole term is indicative of an anomaly under conformal transformation of the boundary metric. As usual in DR, the pole corresponds to the logarithmic divergence in a cutoff regularization. In the pioneering work of Henningson and Skenderis [5], an IR cutoff is used and the anomaly turns out to be the coefficient of the logarithmic divergence after the radial integration is performed.

2.2 Dimensionally regularized one-loop effective potential

As effective potential we understand the integrand in (2.5), i.e.

$$\frac{\partial}{\partial m^2} (V_{d+1}^+ - V_{d+1}^-) = \frac{1}{2} \{G_{\Delta_+}(z, z) - G_{\Delta_-}(z, z)\}, \tag{2.14}$$

where the propagator at coincident points, understood as analytically continued [21] from $D = d - \epsilon$, is given by ($m^2 = \Delta(\Delta - d)$, $\Delta_{\pm} = d/2 \pm \nu$)

$$G_{\Delta}(z, z) = \frac{\Gamma(\Delta)}{2^{1+\Delta} \pi^{\frac{D}{2}} \Gamma(1 + \Delta - \frac{D}{2})} F\left(\frac{\Delta}{2}, \frac{1 + \Delta}{2}; 1 + \Delta - \frac{D}{2}; 1\right). \tag{2.15}$$

Using now Gauß’s formula for the hypergeometric with unit argument and Legendre duplication formula for the gamma function (appendix B), the dimensionally regularized version of (2.14) can be written as

$$\frac{\partial}{\partial m^2} (V_{D+1}^+ - V_{D+1}^-) = \frac{1}{2^{D+2} \pi^{\frac{D+1}{2}}} \Gamma\left(\frac{1 - D}{2}\right) \left[\frac{\Gamma(\nu + \frac{D}{2})}{\Gamma(1 + \nu - \frac{D}{2})} - (\nu \rightarrow -\nu) \right]. \tag{2.16}$$

Letting now $\epsilon \rightarrow 0$, the limit is trivial to take when $d = \text{even}$ since all terms are finite; for $d = \text{odd}$ however, care must be taken to cancel the pole of the gamma function with the zero coming from the expression in square brackets in that case. This is in agreement with general results of QFT in curved space; using heat kernel and dimensional regularization one can show that in odd-dimensional spacetimes the dimensionally regularized effective

potential is finite, whereas in even dimensions the UV singularities show up as a pole at the physical dimension (see, e.g., [22]), which cancel in the difference taken above. Ultimately, one gets a finite result valid for both even and odd d

$$\frac{\partial}{\partial m^2} (V_{d+1}^+ - V_{d+1}^-) = \frac{1}{2\nu} \frac{1}{2^d \pi^{\frac{d}{2}}} \frac{(\nu)^{\frac{d}{2}} (-\nu)^{\frac{d}{2}}}{(\frac{1}{2})^{\frac{d}{2}}} \equiv \mathcal{A}_d(\nu). \quad (2.17)$$

We used the last equation also to introduce an abbreviation $\mathcal{A}_d(\nu)$ for later convenience. That this formula comprises both even and odd d can be better appreciated in the derivation given in appendix A. Written in the form (2.17), this result coincides with that of Gubser and Mitra (eq. 24 in [1]) but now valid for d odd as well. We have to keep in mind to undo the derivative at the end. Interestingly, for the corresponding integral the integrand is essentially the *Plancherel measure*⁴ for the hyperbolic space at imaginary argument $i\nu$ (see appendix A).

2.3 Dimensionally regularized one-loop effective action

The product of the regularized volume (2.9) and the regularized one-loop potential (2.16) yields the dimensionally regularized one-loop effective action

$$\begin{aligned} \frac{\partial}{\partial m^2} (S_{D+1}^+ - S_{D+1}^-) &= \frac{1}{2} \Gamma(-D) \left[\frac{\Gamma(\nu + \frac{D}{2})}{\Gamma(1 + \nu - \frac{D}{2})} - (\nu \rightarrow -\nu) \right] \\ &= \frac{\sin \pi \nu}{2 \sin \pi D/2} \frac{\Gamma(\frac{D}{2} + \nu) \Gamma(\frac{D}{2} - \nu)}{\Gamma(1 + D)}. \end{aligned} \quad (2.18)$$

The poles of $\Gamma(-D)$ are deceiving. For $D \rightarrow \text{odd}$, the pole is canceled against a zero from the square bracket. Only at $D \rightarrow \text{even}$ there is a pole.

The claim now is that this full result can be recovered from the dual boundary theory computation if we use the same regularization procedure, namely dimensional regularization.

3. The boundary computation: deformed partition function

Putting the CFT on the sphere \mathbb{S}^d with radius R , the kernel Ξ becomes [2]

$$\Xi = \delta^d(x, x') + \frac{f}{s^{2\Delta_-}(x, x')}, \quad (3.1)$$

where s is the chordal distance on the sphere. The quotient of the partition functions in the α - and β -theory is then given by

$$W_d^+ - W_d^- \equiv -\log \frac{Z_\beta}{Z_\alpha} = \frac{1}{2} \lim_{f \rightarrow \infty} \log \det \Xi = \lim_{f \rightarrow \infty} \frac{1}{2} \sum_{l=0}^{\infty} \text{deg}(d, l) \log(1 + f g_l), \quad (3.2)$$

⁴Presumably, the easiest way to see this is via the spectral representation in terms of spherical functions (see e.g. [23]), it picks up the residue at $i\nu$. But the construction is valid only for the Δ_+ propagator, Δ_- is only reached at the end by suitable continuation. These details will be presented elsewhere.

where

$$g_l = \pi^{\frac{d}{2}} 2^{2\nu} \frac{\Gamma(\nu)}{\Gamma(\frac{d}{2} - \nu)} \frac{\Gamma(l + \frac{d}{2} - \nu)}{\Gamma(l + \frac{d}{2} + \nu)} R^{2\nu}, \quad (3.3)$$

$$\begin{aligned} \text{deg}(d, l) &= (2l + d - 1) \frac{(l + d - 2)!}{l! (d - 1)!} \\ &\equiv \frac{2l + d - 1}{d - 1} \frac{(d - 1)_l}{l!}. \end{aligned} \quad (3.4)$$

Here g_l is the coefficient⁵ of the expansion of $s^{-2\Delta}(x, x')$ in spherical harmonic and $\text{deg}(d, l)$ counts the degeneracies. For large l one finds (our interest concerns $0 \leq \nu < 1$, see (1.1))

$$\text{deg}(D, l) \propto l^{D-1}, \quad g_l \propto l^{-2\nu}, \quad (3.5)$$

implying convergence of the sum in (3.2) for $D < 2\nu$. To define (3.2) for the physically interesting positive integers d we favor dimensional regularization and use analytical continuation from the save region $D < 0$. There, in addition, the limit $f \rightarrow \infty$ can be taken under the sum.

For this limit an amusing property of the sum of the degeneracies $\text{deg}(D, l)$ alone turns out to be crucial. After short manipulations it can be casted into the binomial expansion of $(1 - 1)^{-D}$ (see (B.6)) which is zero for negative D , i.e.

$$\sum_{l=0}^{\infty} \text{deg}(D, l) = 0. \quad (3.6)$$

As a consequence all factors in g_l not depending on l have no influence on the limit $f \rightarrow \infty$ and we arrive at

$$W_D^+ - W_D^- = -\frac{1}{2} \sum_{l=0}^{\infty} \text{deg}(D, l) \log \frac{\Gamma(l + \frac{D}{2} + \nu)}{\Gamma(l + \frac{D}{2} - \nu)}. \quad (3.7)$$

Here we want to stress that this is our full answer, whereas it is just a piece in [2] where zeta-function regularization was preferred. Although the zeta-function regularization of the sum of the degeneracies alone vanish in odd dimensions, in even dimension it is certainly nonzero.

To make contact with the mass derivative of the effective action of the previous section we take the derivative $\frac{\partial}{\partial m^2} = \frac{1}{2\nu} \frac{\partial}{\partial \nu}$

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_D^+ - W_D^-) = -\frac{1}{4\nu} \sum_{l=0}^{\infty} \text{deg}(D, l) \left(\psi \left(l + \frac{D}{2} + \nu \right) + \psi \left(l + \frac{D}{2} - \nu \right) \right). \quad (3.8)$$

The task is now to compute the sum. For this we want to exploit Gauß's integral representation for $\psi(z)$ (B.5). However, since it requires $z > 0$ we first keep untouched the $l = 0$ term and get for $2\nu - 2 < D < 0$

$$\begin{aligned} \frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_D^+ - W_D^-) &= -\frac{1}{4\nu} \left(\psi \left(\frac{D}{2} + \nu \right) + \psi \left(\frac{D}{2} - \nu \right) \right) \\ &\quad - \frac{1}{4\nu} \sum_{l=1}^{\infty} \text{deg}(D, l) \int_0^{\infty} dt \left(2 \frac{e^{-t}}{t} - \frac{e^{-t(l+d/2)}}{1 - e^{-t}} (e^{-t\nu} + e^{t\nu}) \right). \end{aligned} \quad (3.9)$$

⁵There is a missing factor of $2^{-\Delta}$ in eqs. (19) and (24) of [2]. It can be traced back to the chordal distance in term of the azimuthal angle $s^2 = 2(1 - \cos\theta)$.

Now the sum of the l independent term under the integral can be performed with (3.6). The other sums via (B.6) can be reduced to $\sum_{l=1}^{\infty} \frac{(D-1)_l}{l!} e^{-tl} = (1 - e^{-t})^{1-D} - 1$. Then with $\psi(z) = \psi(1+z) - 1/z$ and the Gauß representation for $\psi(D/2 + 1 \pm \nu)$ we arrive at

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_D^+ - W_D^-) = \frac{D}{\nu(D-2\nu)(D+2\nu)} + \frac{1}{4\nu} \int_0^1 du u^{\frac{D}{2}-1} (u^\nu + u^{-\nu}) \left((1-u)^{-D-1} (1+u) - 1 \right). \tag{3.10}$$

In identifying the remaining integral as a sum of Beta functions a bit of caution is necessary since we are still confined to the convergence region $2\nu - 2 < D < 0$. But using both the standard representation and the subtracted version (B.7) we finally get

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_D^+ - W_D^-) = \frac{1}{2} \Gamma(-D) \left[\frac{\Gamma(\nu + \frac{D}{2})}{\Gamma(1 + \nu - \frac{D}{2})} - (\nu \rightarrow -\nu) \right]. \tag{3.11}$$

This has been derived by allowed manipulations of convergent sums and integrals in the region $2\nu - 2 < D < 0$. From there we analytically continue and a comparison with (2.18) gives now for all D

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_D^+ - W_D^-) = \frac{\partial}{\partial m^2} (S_{D+1}^+ - S_{D+1}^-). \tag{3.12}$$

Let us just mention that one can, in principle, choose Hadamard regularization, i.e. subtract as many terms of the Taylor expansion in ν of the sum (3.7) as necessary to guarantee convergence. This results in the renormalized bulk result plus a polynomial in ν of degree d . This polynomial is just an artifact of the regularization scheme and is of no physical meaning. The question whether in this framework there is a subtraction scheme on the boundary that exactly reproduces the bulk result seems to find an answer in a generalization of Weierstrass formula for the multigamma functions [24]. Surprisingly, the effective potential in AdS can be written in terms the multigamma functions [21, 25]. We refrain from pursuing this *Weierstrass regularization* here and stick to DR for simplicity.

4. Back to the physical dimensions

Let us now send $\epsilon \rightarrow 0$ in the dimensionally regularized partition functions (eqs. 2.18 and 3.12) and see what happens in odd and even dimensions.

4.1 d=odd: renormalized partition functions

Let us assume a minimal subtraction scheme to renormalize and establish the holographic interpretation of the boundary result. In this case we have

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_D^+ - W_D^-) = \frac{\pi}{2\nu} \frac{(-1)^{\frac{d+1}{2}}}{\Gamma(1+d)} (\nu)_{\frac{d}{2}} (-\nu)_{\frac{d}{2}} + o(1). \tag{4.1}$$

Now, the renormalized value is exactly the renormalized volume times the renormalized effective potential⁶

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_d^+ - W_d^-) = \mathcal{V}_{d+1} \cdot \mathcal{A}_d(\nu) = \frac{\partial}{\partial m^2} (S_{d+1}^+ - S_{d+1}^-) . \quad (4.2)$$

This completes the matching in [2], the finite nonzero bulk result being indeed a finite contribution in the CFT computation which has not been computed before. At the same time, it is not a contribution to the conformal anomaly, this being absent for d odd as expected on general grounds.

So, holography (AdS/CFT) in this case matches the renormalized partition functions at $O(1)$ order in CFT_d and at one-loop quantum level in AdS_{d+1} .

4.2 d =even: anomaly and renormalized partition functions

Following the same steps as above, we get for this case

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_D^+ - W_D^-) = \frac{1}{\epsilon} \cdot \frac{1}{\nu} \frac{(-1)^{\frac{d}{2}}}{\Gamma(1+d)} (\nu)^{\frac{d}{2}} (-\nu)^{\frac{d}{2}} + \frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_d^+ - W_d^-) + o(1). \quad (4.3)$$

Here we can identify the factorized form of the term containing the pole, the contribution to the conformal anomaly,

$$\text{Res} \left[\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_D^+ - W_D^-) , D = d \right] = \mathcal{L}_{d+1} \cdot \mathcal{A}_d(\nu) . \quad (4.4)$$

Note that according to (2.17) $\mathcal{A}_d(\nu)$ is just the derivative of the difference of the renormalized effective potentials for the α - and β -CFT.

This is the proof to generic even dimension of the matching between bulk [1] and boundary [2] computations concerning the correction to the *conformal anomaly*, including the overall coefficient.

However, there is apparently a puzzle here concerning the finite remnant. The renormalized value

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_d^+ - W_d^-) = \frac{\mathcal{L}_{d+1} \cdot \mathcal{A}_d(\nu)}{2} \left\{ 2\psi(1+d) - \psi\left(\frac{d}{2} + \nu\right) - \psi\left(\frac{d}{2} - \nu\right) \right\} \quad (4.5)$$

is certainly non-polynomial in ν .

Had we computed only the renormalized effective potential, then after subtraction of the pole we would end up with the finite result $\mathcal{V}_{d+1} \cdot \mathcal{A}_d(\nu)$. But $\mathcal{A}_d(\nu)$ is polynomial in ν and therefore it could have been renormalized away. Yet, the CFT computation renders the non-polynomial finite result of above that cannot be accounted for by the renormalized effective potential, which is only polynomial in ν .

Here is that IR-UV connection enters in a crucial way, and the non-polynomial result is obtained by the cancellation of the pole term in the regularized volume (IR) with the $O(\epsilon)$

⁶This exact agreement can be upset if different regularization/renormalization procedures were chosen, but in any case this ambiguity would show up only as a polynomial in ν of degree d at most. This is related to the fact that if one differentiates enough times with respect to ν (equivalently, m^2) the result is no longer divergent and therefore “reg.-scheme”-independent.

term in the regularized effective potential (UV). Only in this way is the naive factorization bypassed. In fact, one can check that the coefficient of the non-polynomial part in the CFT computation is precisely the \mathcal{L} factor, rather than the regularized volume \mathcal{V} .

That is, we have to keep track of the $O(\epsilon)$ term in the expansion of the regularized effective potential (2.16), (2.17)

$$\frac{\partial}{\partial m^2} (V_{D+1}^+ - V_{D+1}^-) = \mathcal{A}_d(\nu) + \epsilon \cdot \mathcal{B}_d(\nu) + o(\epsilon), \quad (4.6)$$

where

$$\mathcal{B}_d(\nu) = \frac{\mathcal{A}_d(\nu)}{2} \left\{ \log(4\pi) + \psi\left(\frac{1}{2} - \frac{d}{2}\right) - \psi\left(\frac{d}{2} + \nu\right) - \psi\left(\frac{d}{2} - \nu\right) \right\}, \quad (4.7)$$

is almost the non-polynomial part of above.

After using two identities for $d = \text{even}$, $\psi(\frac{1}{2} - \frac{d}{2}) = \psi(\frac{1}{2} + \frac{d}{2})$ and then $2\psi(1 + d) = 2 \log 2 + \psi(\frac{1}{2} + \frac{d}{2}) + \psi(1 + \frac{d}{2})$, which are the “log-derivatives” of Euler’s reflection and Legendre duplication formula respectively (B.1), (B.3), one can finally write the renormalized CFT_d result (4.5) in terms of the bulk quantities (2.12), (4.7) for $d = \text{even}$ as

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} (W_d^+ - W_d^-) = \mathcal{V}_{d+1} \cdot \mathcal{A}_d(\nu) + \mathcal{L}_{d+1} \cdot \mathcal{B}_d(\nu) = \frac{\partial}{\partial m^2} (S_{d+1}^+ - S_{d+1}^-). \quad (4.8)$$

5. Miscellaneous comments

Our main results (3.12), (4.2), (4.8) still contain a mass derivative (equivalently, derivative with respect to ν). Integrating these equations introduces an integration constant which cannot be fixed without further input. Equivalently, so far we only know (see (3.2))

$$W_d(\nu) - W_d(\nu_0) = -\log \left(\frac{Z_\beta(\nu)}{Z_\alpha(\nu)} \frac{Z_\alpha(\nu_0)}{Z_\beta(\nu_0)} \right). \quad (5.1)$$

Beyond dimensional regularization, in the framework of general renormalization theory, there appear free polynomials in ν anyway. Hence fixing this constant should be part of the physically motivated normalization conditions.

It was argued in [1] that both Z_α and Z_β at the BF mass, i.e. $\nu = 0$, should coincide; the argument given was shown in [3] to apply to the vacuum energy rather than to the effective potential and the equality was argued in a different way, replacing the BF mass by infinity as a reference point. We just want to point out that this procedure also has a potential loophole, namely the integration range exceeds the window in which the two CFTs are defined $m_{\text{BF}}^2 \leq m^2 < 1 + m_{\text{BF}}^2$.

Drawing attention by the last comment to the case $\nu = 0$, another remark is in order. Then in (3.2) the product $f g_l$ is ill defined if f is assumed to be ν -independent, as in [2, 3]. However, if one chooses

$$\tilde{f} = f \pi^{\frac{d}{2}} \frac{\Gamma(1 - \nu)}{\nu \Gamma(\frac{d}{2} - \nu)} \quad (5.2)$$

as the true ν -independent quantity, then the product $f(\nu) g_l(\nu) \equiv \tilde{f} k_l(\nu)$ is well defined at $\nu = 0$. The relative factor between f and \tilde{f} can be traced back to the conventional

normalization of the two point functions. In addition, while f is the coefficient of the relevant perturbation of the α -CFT, \tilde{f} appears in the parametrization of the boundary behavior of the bulk theory [2, 3]. However, fortunately, a switch from f to \tilde{f} has no effect on the limit $f \rightarrow \infty$ in (3.2) and the conclusions drawn from it in the previous section. This follows from the observation stated after eq. (3.6): any rescaling of f by a factor independent of l does not affect the limit.

The issue of the integration constant discussed above leading to (5.1) has still another aspect concerning the treatment of (3.2). Starting from the formal expression for $W_d(\nu) - W_d(\nu_0)$ on the r.h.s. we would get $\log(\frac{1+f g_l(\nu)}{1+f g_l(\nu_0)})$ instead of $\log(1 + f g_l(\nu))$. Now the limit $f \rightarrow \infty$ is well defined for each l . The definition of the regularized sum over l could then be done directly with the summands referring to the difference of the two limiting conformal theories. Finally, differentiation with respect to ν would reproduce all our results of section 3.

Recently, in ref. [3] a “kinematic explanation” of the equivalence of the bulk and boundary computation has been given. There a polar basis in \mathbb{H}^{d+1} was used to compute the bulk fluctuation determinants. After inserting the eigenvalues of the angular Laplacian one ends up with a sum over spherical harmonics of effective radial determinants which are now one-dimensional. Using then a proposed generalization of the Gel’fand-Yaglom formula [26], it results in the same expansion as obtained on the boundary (3.2). Since, especially in their reasoning, it should be crucial to have a well defined limit $f \rightarrow \infty$ before the sum over l is taken, we would prefer to consider the cross-ratios

$$\frac{\det_{\tilde{f}_1}(-\Delta_{\text{rad}} + \mathcal{V}_{\text{eff}}(\nu))}{\det_{\tilde{f}_1}(-\Delta_{\text{rad}} + \mathcal{V}_{\text{eff}}(\nu_0))} \cdot \frac{\det_{\tilde{f}_2}(-\Delta_{\text{rad}} + \mathcal{V}_{\text{eff}}(\nu_0))}{\det_{\tilde{f}_2}(-\Delta_{\text{rad}} + \mathcal{V}_{\text{eff}}(\nu))} \tag{5.3}$$

instead of the single ratios obtained by dropping the ν_0 determinants. Besides giving a well defined limit for $\tilde{f}_1 \rightarrow \infty$, $\tilde{f}_2 \rightarrow 0$ this has the additional benefit that no generalization of the Gel’fand-Yaglom formula beyond that in [27] is needed to handle the ratio for operators with different boundary conditions; each of the two ratios in (5.3) refer to the same boundary condition. Even though the above recipe makes finite the quotient of the effective radial determinants, the inclusion of the infinite tower of harmonics makes the sum divergent. This remaining divergence is then the only source for IR divergence in the bulk and UV ones on the boundary. The *formal* equality calls for a more ambitious program including generic dimension and not only the matching of the anomalous part. There is nothing in the derivation that picks out $d = \text{even}$ in preference to $d = \text{odd}$. What we have shown in the previous section is that the equality can indeed be made rigorous if interpreted in the sense of dimensional regularization.

6. Conclusion

The relevant double-trace deformation of a CFT and its AdS dual picture provide a satisfactory test of the correspondence. The regimes in which the bulk and boundary computations are legitimate fully overlap and the mapping goes beyond the original correction to the conformal anomaly. Rather the full change in the partition functions is correctly reproduced

on either side of the correspondence; on the boundary being subleading $O(1)$ in the large N limit, and in the bulk being a 1-loop quantum correction to the classical gravity action. Dimensional regularization proved to be the simplest and most transparent way to control the divergences on both sides, UV and IR infinities are then on equal footing, in accord with the IR-UV connection.

The anomaly turns out to be the same computed in DR and in zeta-function regularization, confirming its stability with respect to changes in the regularization method used [22]. They differ, however, in the regularized value of the boundary determinant for even dimension; this is also known to be the case in free CFTs in curved backgrounds when computing the regularized effective potential [22]. Going back to the anomaly, we recall that it arises in DR due to a cancellation of the pole against a zero, in fact minus the variation of the counterterms is the variation of the renormalized effective action [20, 22]. The pole we already had from the volume regularization and the zero comes from the invariance of \mathcal{L} [17].

At odd d , the finite non-zero bulk change in the effective potential is reproduced from a finite remnant in the boundary computation, confirming the suspicion in [2]. But there is no anomaly in this case, just a conformally invariant renormalized contribution to the partition functions. At even d , in turn, the boundary change in the partition function is obtained only after a subtle cancellation of the pole in the regularized volume (IR-div.) against a zero from the change in the effective potential (UV-div.). This mixing of IR and UV effects on the same side of the correspondence has no precedent in the leading order computation [5]. In that case the bulk computation is a tree level one, where no UV problems show up; i.e. the AdS answer is obtained from the classical SUGRA action.

We can contemplate several extensions of the program carried out. One can try to access to an intermediate stage of the RG flow, that is, finite f . Being away from conformality, the factorization of the volume breaks down, the propagators at coincidence points depend on the radial position; this makes the task of regularization more difficult. Extensions to other bulk geometries seems, naively, immediate in terms of the Plancherel measure, it admits a readily generalization to symmetric spaces [28]. It would be interesting to explore whether this construction admits a holographic interpretation. In the other direction, one can trade the round sphere by a “squashed” one, conformal boundary of Taub-Nut-AdS and Taub-Bolt-AdS spacetimes.

Finally, on the basis of the impressive agreement, one may wonder whether there is a parallel computation in the mathematical literature. If one is willing to allow for a continuation in Δ_- so that it becomes $d/2 - k$ ($k = 1, 2, \dots, d/2$), then $\Xi \sim \langle O_\alpha O_\alpha \rangle \sim 1/s^{2\Delta_-}$ can be thought of as the inverse of the k -th GJMS conformal Laplacian [29]. These are conformally covariant differential operators whose symbol is given the k -th power of the Laplacian; for $k = 1$ one has just the conformal Laplacian (Yamabe operator), $k = 2$ corresponds to the Paneitz operator, etc. For even $d + 1$, we find then an analogous result in theorem 1.4 of [30] for a generalized notion of determinant of the k -th GJMS conformal Laplacian. The absence of anomaly for odd d is consistent with this determinant being a conformal invariant of the conformal infinity of the even dimensional asymptotically hyperbolic manifold. Unfortunately, “the delicate case of $d + 1$ odd where things do not

renormalize correctly”, is still to be understood in this mathematical setting. We anticipate, by analogy with our results, that a proper analysis in this case should unravel a conformal anomaly which can be read off from quotient formulas (generalized Polyakov formulas, see e.g. [32]) of determinants of GJMS operators at conformally related metrics, which involve Q-curvature.

We expect the AdS/CFT recipe to treat double-trace deformations and its bulk interpretation to be a way into these constructions in conformal geometry.⁷ The leading large-N anomaly matching already hinted in this direction, the relation between Q-curvature and volume renormalization emerges there; but higher-dimensional Q-curvatures in connection with generalized Polyakov formulas for GJMS operators had not shown up so far.

Acknowledgments

We have benefited from common seminars with the group of Differential Geometry and Global Analysis at HU-Berlin, in particular those given by J. Erdmenger and A. Juhl. We also thank A. Wipf for sharing his insight on Weierstrass regularization and H. S. Yang for discussions. This work was partially supported by DFG via DO 447/4-1 and IMPRS for *Geometric Analysis, Gravitation and String Theory*.

A. Convolution of bulk-to-boundary propagators

The bulk-to-bulk propagator and bulk-to-boundary propagators in AdS (see, e.g., [10]), in Poincare coordinates

$$ds^2 = \frac{1}{z_0^2}(dz_0^2 + d\vec{x}^2), \tag{A.1}$$

are given, respectively, by

$$G_\Delta(z, w) = C_\Delta \frac{2^{-\Delta}}{2\Delta - d} \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta + 1}{2}; \Delta - \frac{d}{2} + 1; \xi^{-2}\right), \tag{A.2}$$

in term of the hypergeometric function, and

$$K_\Delta(z, \vec{x}) = C_\Delta \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2}\right)^\Delta, \tag{A.3}$$

with the normalization constant

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})}. \tag{A.4}$$

The quantity

$$\xi = \frac{2 z_0 w_0}{z_0^2 + w_0^2 + (\vec{z} - \vec{w})^2} \tag{A.5}$$

is related to the geodesic distance $d(z, w) = \log \frac{1 + \sqrt{1 - \xi^2}}{\xi}$.

⁷GJMS operators have already appeared in matter contributions to the conformal anomaly [31].

There is a natural way to get precisely the difference of the two bulk-to-bulk propagators of conjugate dimension in AdS/CFT. It is based on the observation that the convolution along a common boundary point of two bulk-to-boundary propagator of conjugate dimensions, that is Δ and $d - \Delta$, results in the difference of the corresponding two bulk-to-bulk propagators [3, 33]. More precisely, the result is in fact the sum

$$\int_{\mathbb{R}^d} d^d \vec{x} K_{\Delta}(z, \vec{x}) K_{d-\Delta}(w, \vec{x}) = (2\Delta - d)G_{\Delta}(z, w) + [\Delta \leftrightarrow d - \Delta]. \quad (\text{A.6})$$

The coincidence limit $w \rightarrow z$ can be taken before the convolution, on both bulk-to-boundary propagators, to get

$$(2\Delta - d) \{G_{\Delta}(z, z) - G_{d-\Delta}(z, z)\} = C_{\Delta} C_{d-\Delta} \int_{\mathbb{R}^d} d^d \vec{x} \frac{z_0^d}{[z_0^2 + (\vec{z} - \vec{x})^2]^d}. \quad (\text{A.7})$$

The z_0 dependence in the integral is just illusory, the result is just $\frac{2\pi^{\frac{d}{2}}}{2^d \Gamma(\frac{d+1}{2})}$. As noted by Dobrev [34], the product of the two bulk-to-boundary normalization factors $C_{\Delta} C_{d-\Delta}$ coincides, modulo factors independent of $\Delta = \frac{d}{2} + \nu$, with the Plancherel measure for the $d+1$ -dimensional hyperbolic space evaluated at imaginary argument $i\nu$. After putting all together, equation (2.17) is confirmed.

B. Useful formulas

Here we collect some formulas that have been used throughout the paper. They can all be found e.g. in [35].

$$\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \quad (\text{Euler's reflection}) \quad (\text{B.1})$$

$$(z)_n = \frac{\Gamma(z + n)}{\Gamma(z)} \quad (\text{Pochhammer symbol})$$

$$(1 + z)_{-n} = \frac{(-1)^n}{(-z)_n} \quad (\text{B.2})$$

$$\Gamma(z) \Gamma(z + \frac{1}{2}) = 2^{1-2z} \Gamma(\frac{1}{2}) \Gamma(2z) \quad (\text{Legendre duplication}) \quad (\text{B.3})$$

$$F(a, b; c; 1) = \frac{(c - b)_{-a}}{(c)_{-a}} \quad (\text{Gauss's hypergeometric theorem, } \text{Re}(c - a - b) > 0) \quad (\text{B.4})$$

$$\psi(z) = \int_0^{\infty} dt \left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{1 - e^{-t}} \right) \quad (\text{Gauss's integral representation}) \quad (\text{B.5})$$

(Binomial expansion)

$$(1-x)^a = \sum_{n=0}^{\infty} \frac{(-a)_n}{n!} x^n \quad (\text{B.6})$$

$$B(a,b) - B(a,c) = \int_0^1 du (1-u)^{a-1} (u^{b-1} - u^{c-1}), \quad a > -1, \quad b, c > 0. \quad (\text{B.7})$$

References

- [1] S.S. Gubser and I. Mitra, *Double-trace operators and one-loop vacuum energy in AdS/CFT*, *Phys. Rev. D* **67** (2003) 064018 [[hep-th/0210093](#)].
- [2] S.S. Gubser and I.R. Klebanov, *A universal result on central charges in the presence of double-trace deformations*, *Nucl. Phys. B* **656** (2003) 23 [[hep-th/0212138](#)].
- [3] T. Hartman and L. Rastelli, *Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT*, [hep-th/0602106](#).
- [4] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [5] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)]; *Holography and the Weyl anomaly*, *Fortschr. Phys.* **48** (2000) 125 [[hep-th/9812032](#)].
- [6] O. Aharony, J. Pawelczyk, S. Theisen and S. Yankielowicz, *A note on anomalies in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 066001 [[hep-th/9901134](#)];
M. Blau, K.S. Narain and E. Gava, *On subleading contributions to the AdS/CFT trace anomaly*, *JHEP* **09** (1999) 018 [[hep-th/9904179](#)];
S.G. Naculich, H.J. Schnitzer and N. Wyllard, *1/N corrections to anomalies and the AdS/CFT correspondence for orientifolded $N = 2$ orbifold models and $N = 1$ conifold models*, *Int. J. Mod. Phys. A* **17** (2002) 2567 [[hep-th/0106020](#)];
S. Nojiri and S.D. Odintsov, *On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence*, *Int. J. Mod. Phys. A* **15** (2000) 413 [[hep-th/9903033](#)].
- [7] A. Bilal and C.-S. Chu, *A note on the chiral anomaly in the AdS/CFT correspondence and $1/N^2$ correction*, *Nucl. Phys. B* **562** (1999) 181 [[hep-th/9907106](#)];
P. Mansfield, D. Nolland and T. Ueno, *Order $1/N^2$ test of the maldacena conjecture. II: the full bulk one-loop contribution to the boundary Weyl anomaly*, *Phys. Lett. B* **565** (2003) 207 [[hep-th/0208135](#)].
- [8] S.S. Gubser, *Loop effects in AdS/CFT and beyond*, *Prog. Theor. Phys. Suppl.* **152** (2004) 30.
- [9] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Ann. Phys. (NY)* **144** (1982) 249.
- [10] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys. B* **556** (1999) 89 [[hep-th/9905104](#)].

- [11] E. Witten, *Multi-trace operators, boundary conditions and AdS/CFT correspondence*, [hep-th/0112258](#);
M. Berkooz, A. Sever and A. Shomer, *Double-trace deformations, boundary conditions and spacetime singularities*, *JHEP* **05** (2002) 034 [[hep-th/0112264](#)];
W. Muck, *An improved correspondence formula for AdS/CFT with multi-trace operators*, *Phys. Lett. B* **531** (2002) 301 [[hep-th/0201100](#)];
P. Minces, *Multi-trace operators and the generalized AdS/CFT prescription*, *Phys. Rev. D* **68** (2003) 024027 [[hep-th/0201172](#)].
- [12] I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical $O(N)$ vector model*, *Phys. Lett. B* **550** (2002) 213 [[hep-th/0210114](#)].
- [13] D.E. Diaz and H. Dorn, *On the AdS higher spin/ $O(N)$ vector model correspondence: degeneracy of the holographic image*, *JHEP* **07** (2006) 022 [[hep-th/0603084](#)].
- [14] G.V. Dunne and K. Kirsten, *Functional determinants for radial operators*, *J. Phys. A* **39** (2006) 1191 [[hep-th/0607066](#)].
- [15] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, [hep-th/9805114](#).
- [16] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [[hep-th/0002230](#)];
M. Bianchi, D.Z. Freedman and K. Skenderis, *Holographic renormalization*, *Nucl. Phys. B* **631** (2002) 159 [[hep-th/0112119](#)];
K. Skenderis, *Lecture notes on holographic renormalization*, *Class. and Quant. Grav.* **19** (2002) 5849 [[hep-th/0209067](#)].
- [17] C.R. Graham, *Volume and area renormalizations for conformally compact Einstein metrics*, [math.DG/9909042](#).
- [18] P. Albin, *Renormalizing curvature integrals on Poincare-Einstein manifolds*, [math.DG/0504161](#).
- [19] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, *Diffeomorphisms and holographic anomalies*, *Class. and Quant. Grav.* **17** (2000) 1129 [[hep-th/9910267](#)].
- [20] A. Schwimmer and S. Theisen, *Diffeomorphisms, anomalies and the Fefferman-Graham ambiguity*, *JHEP* **08** (2000) 032 [[hep-th/0008082](#)].
- [21] M. Kamela and C.P. Burgess, *Massive-scalar effective actions on anti-de Sitter spacetime*, *Can. J. Phys.* **77** (1999) 85 [[hep-th/9808107](#)].
- [22] B.S. DeWitt, *The global approach to quantum field theory*, vol. **2**, Int. Ser. Monogr. Phys. **114** (2003).
- [23] R. Camporesi, *Zeta function regularization of one loop effective potentials in anti-de Sitter space-time*, *Phys. Rev. D* **43** (1991) 3958.
- [24] M. Vigneras, *L'équation fonctionnelle de la fonction zeta de Selberg du groupe modulaire $PSL(2, Z)$* , *Asterisque* **61** (1979) 235.
- [25] A. Das and G.V. Dunne, *Large-order perturbation theory and de Sitter/anti de Sitter effective actions*, *Phys. Rev. D* **74** (2006) 044029 [[hep-th/0607168](#)].
- [26] I.M. Gelfand and A.M. Yaglom, *Integration in functional spaces and its applications in quantum physics*, *J. Math. Phys.* **1** (1960) 48.

- [27] K. Kirsten and A.J. McKane, *Functional determinants for general Sturm-Liouville problems*, *J. Phys. A* **37** (2004) 4649 [[math-ph/0403050](#)].
- [28] S. Helgason, *Geometric analysis on symmetric spaces*, Math. Surveys and Monogr. **39**, Amer. Math. Soc. (1994).
- [29] C.R. Graham, R. Jenne, L.J. Mason and G.A.J. Sparling, *Conformally invariant powers of the laplacian, I: Existence*, *J. London Math. Soc.* **46** (1992) 557.
- [30] C. Guillarmou, *Generalized Krein formula, determinants and Selberg Zeta function in even dimension*, [math.SP/0512173](#).
- [31] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, [hep-th/0404176](#).
- [32] T. Branson, *Sharp inequalities, the functional determinant, and the complementary series*, *Trans. Amer. Math. Soc.* **347** (1995) 3671.
- [33] T. Leonhardt, R. Manvelyan and W. Ruhl, *The group approach to AdS space propagators*, *Nucl. Phys. B* **667** (2003) 413 [[hep-th/0305235](#)].
- [34] V.K. Dobrev, *Intertwining operator realization of the AdS/CFT correspondence*, *Nucl. Phys. B* **553** (1999) 559 [[hep-th/9812194](#)].
- [35] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, in *Encyclopedia of Mathematics and Its Applications*, **71**, Cambridge University Press (1999).